Int. J. Heat Mass Transfer. Vol. 16, pp. 2377–2391. Pergamon Press 1973. Printed in Great Britain

LAMINAR HEAT TRANSFER TO TEMPERATURE-DEPENDENT BINGHAM FLUIDS IN TUBES

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(*Received* 5 *March* 1973 *and in revised form* 30 *April* 1973)

Abstract—This paper presents a theoretical treatment of laminar flow heat transfer in circular tubes for a temperature-dependent non-Newtonian fluid for which the relationship between the shear stress, τ, and the shear rate, $\dot{\gamma}$, can be described by an equation of the form

$$\tau = \tau_y + K(T)\dot{\gamma}^n$$

where τ_y is a yield stress, n is a constant and $K(T)$ is a function of temperature. This model can therefore cater for both power-law and Bingham plastic behaviour. The two boundary conditions of constant wall temperature and constant wall heat flux are considered for both heating and cooling situations. The computed results are presented by plotting a Nusselt number as a function of the Graetz number with dimensionless groups specifying the temperature dependence effect, the rheological properties and the wall conditions as parameters. This method of presentation is convenient for engineering design purposes. Temperature profiles, velocity profiles and the pressure drop can also be determined.

NOMENCLATURE

a, tube radius;

Br, Brinkmann number, $\bar{u}^{n+1}K_i/kT_i a^{n-1}$ [dimensionless];

C_p, specific heat;

Gz, Graetz number, WC_p/kx [dimensionless];

$\bar{h}_T$, mean heat transfer coefficient for constant wall temperature;

h_q, local heat transfer coefficient for constant wall heat flux;

k, thermal conductivity;

$K(T)$, function of temperature defined by equation (8);

K_i, value of $K(T)$ at inlet temperature T_i;

n, flow behaviour index [dimensionless];

$\overline{Nu}_T$, mean Nusselt number, $2a\bar{h}_T/k$ [dimensionless];

Nu_q, local Nusselt number, $2ah_q/k$ [dimensionless];

p, pressure;

q_w, wall heat flux;

r, radial distance coordinate;

R, r/a [dimensionless];

T, temperature;

T_i, inlet temperature;

T_w, wall temperature;

T_0, bulk outlet temperature;

u, velocity;

$\bar{u}$, mean velocity;

$\bar{u}_i$, mean velocity at the tube entrance;

$\bar{u}_x$, mean velocity at axial distance, x;

U, $u/\bar{u}$ [dimensionless];

W, mass flow rate;

x, axial distance coordinate;

X, $kx/\rho\bar{u}a^2C_p = \pi/Gz$ [dimensionless].

Greek symbols

β, parameter representing the temperature dependence effect, βT_i [dimensionless];

β_i, constant characterising temperature dependent properties;

$\dot{\gamma}$, shear rate;

ϕ, T_w/T_i [dimensionless];

ψ, $q_w a/kT_i$ [dimensionless];

ρ, fluid density;
τ, shear stress;
τ_w, wall shear stress;
θ, $(T - T_i)/T_i$ [dimensionless];
θ_0, $(\bar{T}_0 - T_i)/T_i$ [dimensionless].

1. INTRODUCTION

LAMINAR heat transfer to non-Newtonian fluids in tubes is a problem of considerable industrial significance and has received much attention in the past.

Asymptotic Nusselt numbers for a variety of non-Newtonian fluids have been derived by Beek and Eggink [1] for both constant wall temperature and constant wall heat flux as boundary conditions. These solutions assume that the rheological properties of the fluid are independent of temperature and are only applicable to the region far removed from the tube entrance where both fully developed velocity and temperature profiles exist. Several other workers have derived solutions for the mean and local Nusselt numbers by assuming that the rheological properties are independent of temperature. For example, power-law fluids have been considered by Lyche and Bird [2] and also by Whiteman and Drake [3], Bingham plastics by Hirai [4] and also by Wissler and Schecter [5], Prandtl–Eyring fluids by Schenk and Van Laar [6]. Pigford [7] and Metzner *et al.* [8, 9] have extended the Leveque approximation to give useful approximate solutions for heat transfer with constant wall temperature for a number of time-independent non-Newtonian fluids.

The restriction that the fluid rheological properties are independent of temperature can be a serious assumption since, in many cases, this effect has a major influence on heat transfer. Several attempts [8–11] have been made to take this effect into account and approximate solutions to the heat transfer problem which utilize an empirical correction factor to account for the temperature dependence of the fluid consistency, have been obtained. Christiansen *et al.* [12, 13] have produced more exact solutions by solving the problem numerically. Their results are presented as a series of graphs of the mean Nusselt number plotted against the Graetz number with dimensionless quantities representing the rheology of the fluid as parameters for the boundary condition of constant wall temperature. Jensen [14] has carried out similar work for ideal Bingham plastics.

The work reported here considers heat transfer to a generalized Bingham plastic in tubes with constant wall temperature or constant wall heat flux. The effect of the temperature dependence of the fluid consistency is taken into account. This type of fluid model also caters for Newtonian behaviour, power-law behaviour and ideal Bingham plastic behaviour. The results of the calculations are presented graphically in terms of dimensionless groups and as such should be convenient for engineering design purposes.

2. FORMULATION OF THE PROBLEM

This work is concerned with heat transfer to non-Newtonian fluids, the rheological behaviour of which can be approximated by a temperature-dependent generalized Bingham plastic model of the form

$$\tau = \tau_y + K(T)\dot{\gamma}^n; \qquad \tau > \tau_y$$

$$\dot{\gamma} = 0; \qquad \tau \leqq \tau_y \qquad (1)$$

where τ is the shear stress, τ_y is the yield stress and is assumed to be independent of temperature, $\dot{\gamma}$ is the shear rate, $K(T)$ is a function of temperature, T, and n is a temperature-independent exponent which is less than unity for shear thinning materials and greater than unity for shear thickening materials. The assumption that the yield stress, τ_y, is independent of temperature was also adopted by Jensen [14] who suggested that this effect is small compared with the temperature dependency of the fluid consistency and can thus be ignored. A possible explanation for this is that the yield stress is mainly dependent on a mechanical locking of the fluid which is essentially temperature independent. It is anticipated that the model will adequately describe the behaviour of many fluids of commercial interest.

The problem of heat transfer in laminar flow in straight tubes will be considered for the two boundary conditions of constant wall temperature and constant wall heat flux which, although idealized situations, are of relevance in the design of much heat transfer equipment.

The following analysis is subject to certain constraints namely:

(a) The flow is laminar and steady.
(b) The fluid heat capacity, C_p, thermal conductivity, k, and density, ρ, are independent of temperature and pressure.
(c) Isothermal flow is fully developed at the entrance to the heated section and the fluid temperature at this point is uniform and constant.
(d) The radial velocity profile within the heated section will change as a result of changes in the rheological properties with temperature but it will be assumed that radial velocities and axial velocity gradients will be small and can be neglected.
(e) Transfer of energy by conduction in the axial direction may be neglected.
(f) Thermal energy generation within the fluid by viscous dissipation or other means is negligible.

Constraints (a)–(e) are not unduly restrictive in practice for many engineering problems. Constraint (f) is relaxed later in the paper.

Equations of motion and energy

With the above assumptions the equation of motion simplifies to

$$-\frac{\partial p}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau) \qquad (2)$$

where $(\partial p/\partial x)$ is the axial pressure gradient and τ is the shear stress at radius r. The corresponding energy equation is

$$\rho C_p u \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad (3)$$

where u and T are the velocity and temperature respectively at radius r for a particular axial distance x.

The equation of motion, i.e. equation (2), can be integrated by taking $(\partial p/\partial x)$ independent of r to give

$$\tau = \frac{r}{a}\tau_w. \qquad (4)$$

If we introduce the following dimensionless quantities

$$\begin{aligned} U &= u/\bar{u} \\ R &= r/a \\ X &= kx/\rho u a^2 C_p = \pi/Gz \\ \theta &= (T - T_i)/T_i \end{aligned} \qquad (5)$$

we get, from equation (3), the energy equation in the following dimensionless form

$$U\frac{\partial \theta}{\partial X} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right). \qquad (6)$$

Also equation (4) becomes

$$\tau = R\tau_w. \qquad (7)$$

Rheological equation for the fluid

Before the problem can be solved it is necessary to specify the form of $K(T)$. The form used in this work is similar to that used by Mizushina and Kuriwaki [10] and has been shown to be reasonable for many materials over the temperature ranges often encountered in industrial processes. It is given by

$$K(T) = \frac{K_i}{\{1 + \beta_i(T - T_i)\}^n} \qquad (8)$$

where K_i is the value of $K(T)$ at the fluid inlet temperature T_i and β_i is a constant which characterizes the temperature dependent properties of the fluid. Hence the rheological behaviour of the material is described by

$$\begin{aligned} \tau &= \tau_y + K_i\left[\frac{\dot{\gamma}}{1 + \beta_i(T - T_i)}\right]^n; \quad \tau > \tau_y \\ \dot{\gamma} &= 0 \qquad ; \quad \tau \leqq \tau_y. \end{aligned} \qquad (9)$$

Equation (6) was solved for the two boundary conditions of constant wall temperature and constant wall heat flux. This is discussed in the following sections.

BOUNDARY CONDITIONS

Constant wall temperature

The boundary conditions for this problem are:

At the tube inlet $\theta = 0$ at $X = 0$ for all R

At the tube wall $\theta = (T_w/T_i - 1)$ (10)

at $R = 1$ for all X

where T_w is the constant wall temperature.

Constant wall heat flux

The boundary conditions for this case are:

At the tube inlet $\theta = 0$ at $X = 0$ for all R.

$$\text{At the tube wall } q_w = \frac{kT_i}{a}\left(\frac{\partial\theta}{\partial R}\right)_w \quad \text{at } R = 1 \text{ for all } X \tag{11}$$

where q_w is the constant wall heat flux.

SOLUTION OF THE PROBLEM

Velocity profile

Substituting for τ in equation (7) and noting that $\dot{\gamma} = -\partial u/\partial r$ gives

$$\tau_y + K_i\left[\frac{-\partial u/\partial r}{1 + \beta_i(T - T_i)}\right]^n = R\tau_w; \quad \tau > \tau_y. \tag{12}$$

Rearranging and putting $dr = a dR$ gives

$$-du = a\left(\frac{\tau_w}{K_i}\right)^{1/n}\left(R - \frac{\tau_y}{\tau_w}\right)^{1/n}\{1 + \beta_i(T - T_i)\}\,dR. \tag{13}$$

Equation (7) can be applied to the radial position at which the yield stress occurs. Thus, if r_y is this radius, equation (7) may be rewritten as

$$R_y = \tau_y/\tau_w \tag{14}$$

where $R_y = r_y/a$.

From equation (5) we get that

$$(T - T_i) = \theta T_i \tag{15}$$

and on substituting equation (14) and (15) into equation (13) we get

$$-du = a\left(\frac{\tau_w}{K_i}\right)^{1/n}(R - R_y)^{1/n}(1 + \beta\theta)\,dR \tag{16}$$

where

$$\beta = \beta_i T_i. \tag{17}$$

Integrating equation (16), assuming no slip at the wall, gives

$$u = a\left(\frac{\tau_w}{K_i}\right)^{1/n}\int_R^1 (1 + \beta\theta)(R - R_y)^{1/n}\,dR; \quad \tau > \tau_y. \tag{18}$$

Equation (18) refers to that part of the fluid which is in shear flow. The portion of the fluid for which $\tau \leqq \tau_y$ flows as a solid plug i.e.

$$u(\text{at } r < r_y) = u(\text{at } r_y). \tag{19}$$

The plug velocity, i.e. the value of u at $R = R_y$ is obtained by carrying out the integration in equation (18) and putting $R = R_y$ into the resulting equation. Equation (18) may be written as

$$u = a\left(\frac{\tau_w}{K_i}\right)^{1/n} I_1 \tag{20}$$

where

$$I_1 = \int_R^1 (1 + \beta\theta)(R - R_y)^{1/n}\,dr; \quad \tau > \tau_y \tag{21}$$

and

$$I_1 = I_1(\text{at } R = R_y); \quad \tau \leqq \tau_y.$$

The mean velocity through the pipe is given by

$$\bar{u} = \frac{2}{a^2}\int_0^a ur\,dr \tag{22}$$

which after substitution for u and R becomes

$$\bar{u} = 2a\left(\frac{\tau_w}{K_i}\right)^{1/n}\int_0^1 I_1 R\,dR. \tag{23}$$

This may be written as

$$\bar{u} = 2a\left(\frac{\tau_w}{K_i}\right)^{\frac{1}{n}} I_2 \tag{24}$$

where

$$I_2 = \int_0^1 I_1 R \, \mathrm{d}R. \tag{25}$$

Hence from equations (20) and (25) we get

$$U = u/\bar{u} = I_1/2I_2. \tag{26}$$

This set of equations for determining the velocity profile must satisfy the equation of continuity. This may be written in terms of the tube inlet conditions and conditions at a point a distance x down the tube as

$$\bar{u}_i = \bar{u}_x \tag{27}$$

where $\bar{u}_i$ is the mean velocity at inlet and $\bar{u}_x$ is the mean velocity at distance x down the tube. It is obvious from equations (21), (24) and (25) that for $\bar{u}$ to remain constant down the heated section as the temperature and hence θ changes, the value of τ_w must alter. By substituting for u from equation (24), equation (27) may be written

$$(\tau_w^{1/n} I_2)_i = (\tau_w^{1/n} I_2)_x. \tag{28}$$

By dividing both sides of equation (28) by $\tau_y^{\frac{1}{n}}$ we get

$$\left(\frac{I_2}{R_y^{\frac{1}{n}}}\right)_i = \left(\frac{I_2}{R_y^{\frac{1}{n}}}\right)_x. \tag{29}$$

The quantity $(I_2/R_y^{\frac{1}{n}})_i$ representing the conditions at the tube inlet is obtained from the isothermal velocity profile and is a constant for a particular value of n and τ_y. Since the value of τ_w changes during heat transfer, then the value of R_y (i.e. τ_y/τ_w) must also change (τ_y being assumed constant for a given problem). Also, since the determination of the velocity profile depends on the value of R_y and vice versa, an iteration procedure was adopted to obtain the value of U at each axial position down the heated section.

Substituting for U in equation (6) gives

$$\frac{I_1}{2I_2}\frac{\partial \theta}{\partial X} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right). \tag{30}$$

SOLUTIONS

The equations were solved numerically to yield solutions as functions of a number of dimensionless parameters, viz:

Constant wall temperature

$$U_{R,X}, \theta_{R,X} = f[Gz, n, (\tau_y/\tau_w)_i, \beta, T_w/T_i] \tag{31}$$

for the boundary conditions described earlier. The quantities n and τ_y are obtained by rheological measurements and the value of τ_w at the tube inlet is calculated for the particular flow rate required. Subscript i refers to conditions at the tube inlet.

(T_w/T_i) is designated ϕ and for heating $\phi > 1$ and for cooling $\phi < 1$.

Constant wall heat flux

$$U_{R,X}, \theta_{R,X} = f[Gz, n, (\tau_y/\tau_w)_i, \beta, (q_w a/kT_i)] \tag{32}$$

for the boundary conditions described earlier. The quantity $(q_w a/kT_i)$ represents the constant wall heat flux boundary condition and is a measure of the dimensionless temperature gradient at the tube wall, i.e. $(\partial\theta/\partial T)_w$. It is designated ψ and for heating $\psi > 0$ and for cooling $\psi < 0$.

The numerical technique used to solve equation (30) is described in detail elsewhere [15]. It consisted of a Crank–Nicholson, Thomas algorithm implicit finite-difference scheme using 100 radial increments and an initial axial step length of 10^{-6}. It was possible to vary these step lengths to allow smaller increments to be used near the tube inlet where changes occur most rapidly and larger step lengths to be used further downstream. At each axial position an iteration procedure based on equation (29) was set up to calculate R_y and hence the velocity profile.

Definition of Nusselt numbers

It is useful to present the results of heat transfer calculations by plotting a Nusselt number against the Graetz number, Gz.

(a) *Constant wall temperature.* For this case a mean Nusselt number can be defined as

$$\overline{Nu_T} = 2a\bar{h}_T/k \quad (33)$$

where h is the heat transfer coefficient, the subscript T indicates constant wall temperature and the bars denote mean values. The heat transfer coefficient $\bar{h}_T$ for a tube of length x is defined in terms of the terminal temperatures as

$$\bar{h}_T = \frac{WC_p(\bar{T}_0 - T_i)}{2\pi ax\{T_w - \frac{1}{2}(T_i + \bar{T}_0)\}} \quad (34)$$

where $\bar{T}_0$ is the bulk outlet (i.e. cup mixing) temperature, and W is the fluid mass flow rate. Further we have that

$$WC_p(\bar{T}_0 - T_i) = 2\pi C_p\rho \int_0^a ru(T - T_i)\,\mathrm{d}r \quad (35)$$

hence

$$\overline{Nu_T} = \frac{2\rho C_p \int_0^a ru(T - T_i)\,\mathrm{d}r}{kx\{T_w - \frac{1}{2}(T_i + \bar{T}_0)\}}. \quad (36)$$

By substituting for θ, R and Gz as given by equation (5) and noting that

$$u = \frac{\bar{u}I_1}{2I_2} = \frac{W}{2\pi a^2\rho}\left(\frac{I_1}{I_2}\right) \quad (37)$$

we finally get

$$\overline{Nu_T} = \frac{Gz}{\pi}\frac{\theta_0}{\theta_w - \frac{1}{2}\theta_0} \quad (38)$$

where θ_0 is the dimensionless bulk outlet temperature, i.e. $\theta_0 = (\bar{T}_0 - T_i)/T_i$, which can be shown to be given by [15]

$$\theta_0 = \int_0^1 R\theta(I_1/I_2)\,\mathrm{d}R. \quad (39)$$

(b) *Constant wall heat flux.* For this case it is useful to define a local Nusselt number as

$$Nu_q = 2ah_q/k \quad (40)$$

where subscript q denotes constant wall heat flux. The local heat transfer coefficient h_q is defined as

$$h_q = k\left(\frac{\partial T}{\partial r}\right)_w \Big/ (T_w - \bar{T}_0) \quad (41)$$

where $(\partial T/\partial r)_w$ is the wall temperature gradient at axial position, x. Thus equation (40) becomes

$$Nu_q = 2a\left(\frac{\partial T}{\partial r}\right)_w \Big/ (T_w - \bar{T}_0)$$

which in terms of the dimensionless quantities described earlier is

$$Nu_q = \frac{2}{\theta_w - \theta_0}\psi. \quad (42)$$

The computed Nusselt numbers as defined by equations (38) and (42) are plotted against the Graetz number for a range of values of the dimensionless parameters, n, $(\tau_y/\tau_w)_i$, β and ϕ or ψ. These parameters describe the particular problem to be investigated and are used as input data for the computer program which was developed [15] to carry out the numerical solutions.

DISCUSSION OF RESULTS

The computed results are shown graphically in Figs. 1–15 and the main features are discussed below. The quantity $(\tau_y/\tau_w)_i$ is designated R_{y_i}.

(a) *Constant wall temperature*

For this case the heat transfer results are presented graphically as plots of $\overline{Nu_T}$ vs Gz.

Fluid consistency independent of temperature. Figure 1 shows the effect of the parameter, R_{y_i}, i.e. $(\tau_y/\tau_w)_i$, on heat transfer to fluids with $n = 1{\cdot}0$ (i.e. Bingham plastic materials). Since the fluid consistency is constant the fluid velocity profile remains constant at its isothermal value during heat transfer. Isothermal velocity profiles for $n = 1{\cdot}0$ and $R_{y_i} = 0, 0{\cdot}5$ and $0{\cdot}7$ are shown in Fig. 2. It can be seen that for a given value of n, as R_{y_i} increases, the velocity profile becomes flatter, i.e. velocity gradients are increased in the tube wall region and decreased near the tube centre. This increase in the velocity gradient

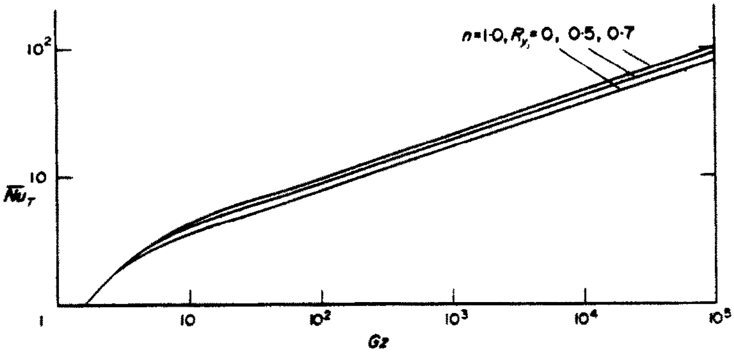

FIG. 1. $\overline{Nu}_T$ vs Gz for the heating and cooling of temperature-independent Bingham plastics with constant temperature at the tube wall.

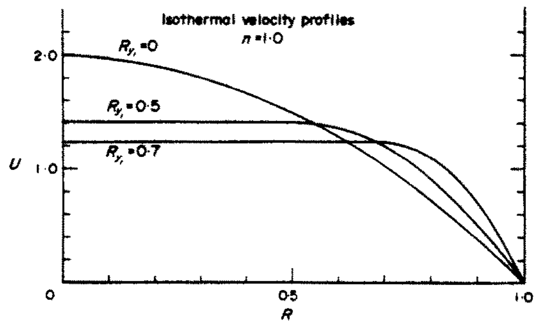

FIG. 2. Isothermal velocity profiles for Bingham plastics in laminar flow.

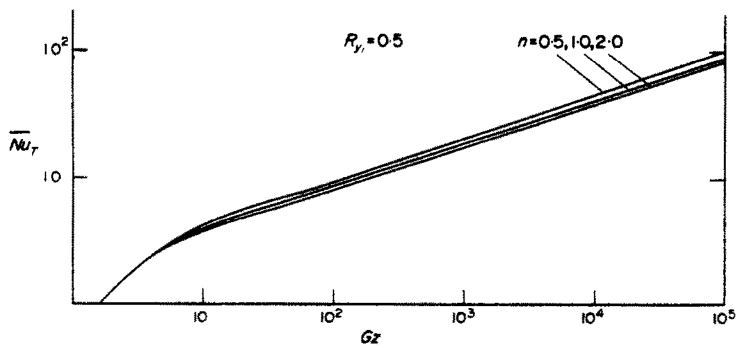

FIG. 3. $\overline{Nu}_T$ vs Gz for the heating and cooling of temperature-independent generalized Bingham plastics with constant temperature at the tube wall.

near the tube wall enhances the heat transfer rate as shown by the increase in $\overline{Nu}_T$ at a given value of Gz as R_{y_i} increases, in Fig. 1.

Figure 3 shows the effect of n on heat transfer for a value of $R_{y_i} = 0{\cdot}5$. The isothermal velocity profiles for a number of values of n with $R_{y_i} = 0{\cdot}5$ are shown in Fig. 4. As can be seen, as n decreases

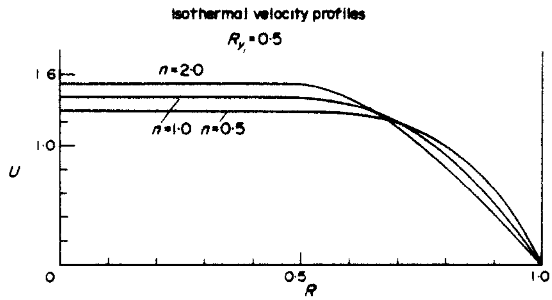

FIG. 4. Isothermal velocity profiles for generalized Bingham plastics in laminar flow.

for a given value of R_{y_i} the velocity gradients are again increased in the wall region leading to an increase in the heat transfer rate as shown in Fig. 3.

Fluid consistency dependent on temperature. For this situation the velocity profile changes during heat transfer. For heating, i.e. $\phi > 1$, the increase in temperature in the wall region decreases the fluid consistency here. This leads to increased velocities near the tube wall. Also for situations where $R_{y_i} > 0$, i.e. where a section of fluid near the tube centre flows as a solid plug, since the velocity profile is changing the value of τ_w and hence R_y must also be changing. For heating, although the shear rate, $\dot{\gamma}$, (i.e. the velocity gradient) increases near the wall during heat transfer, the constraint of constant wall temperature means that the shear rate actually at the wall decreases. Thus it can be seen from equation (1) that the wall shear stress decreases and so the value of R_y, which represents the size of the solid flowing plug, increases during heating (i.e. R_y increases as

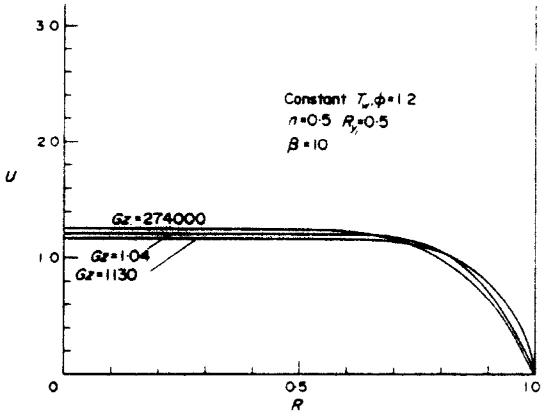

FIG. 5. Development of the velocity profile during the heating of a temperature-dependent generalized Bingham plastic ($n = 0{\cdot}5$, $\tau_y/\tau_w = 0{\cdot}5$) with constant temperature at the tube wall.

Gz decreases). This is shown in Fig. 5. Far downstream, i.e. where Gz is small the temperature becomes uniform across the tube at a value approaching T_w, and the velocity profile is then fully developed. If the fluid had had no yield stress, i.e. τ_y and hence (τ_y/τ_w) is equal to zero, the velocity profile would have reverted back to its isothermal value when the temperature became uniform across the tube. However, the

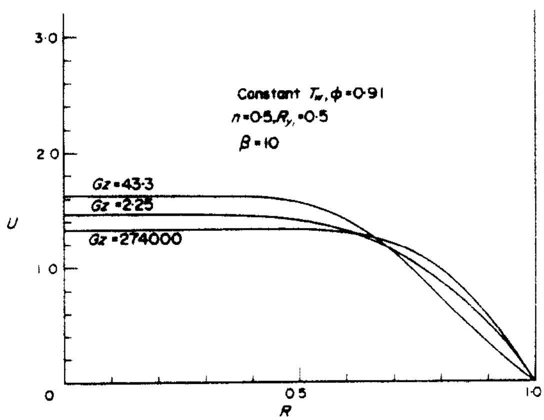

FIG. 6. Development of the velocity profile for the cooling of a temperature-dependent generalized Bingham plastic ($n = 0{\cdot}5$, $\tau_y/\tau_w = 0{\cdot}5$) with constant temperature at the tube wall.

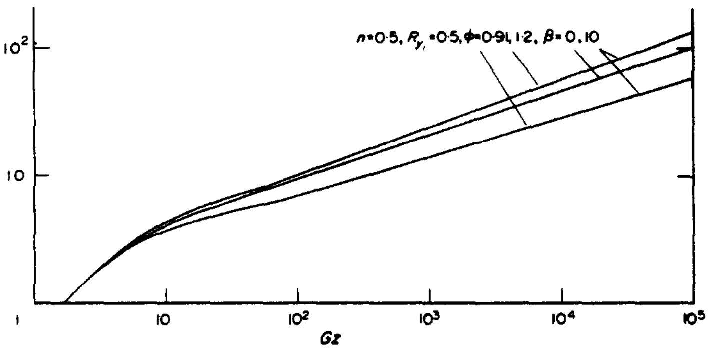

FIG. 7. $\overline{Nu}_T$ vs Gz for the heating and cooling of a temperature-dependent generalized Bingham plastic ($n = 0{\cdot}5, \tau_y/\tau_w = 0{\cdot}5$) with constant temperature at the tube wall.

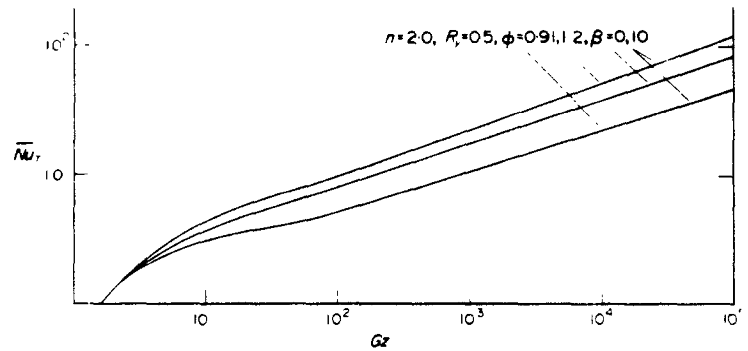

FIG. 8. $\overline{Nu}_T$ vs Gz for the heating and cooling of a temperature-dependent generalized Bingham plastic ($n = 2{\cdot}0, \tau_y/\tau_w = 0{\cdot}5$) with constant temperature at the tube wall.

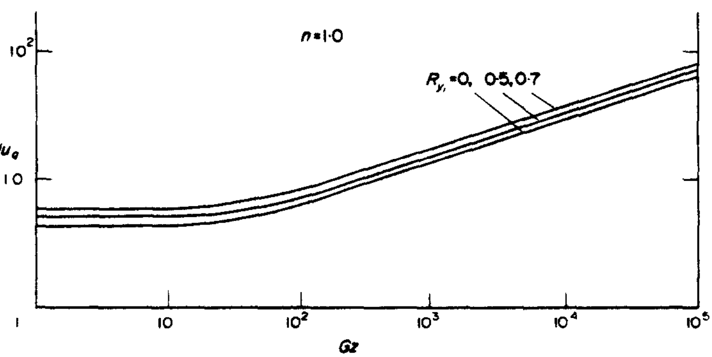

FIG. 9. Nu_q vs Gz for the heating and cooling of temperature-independent Bingham plastic with constant heat flux at the tube wall.

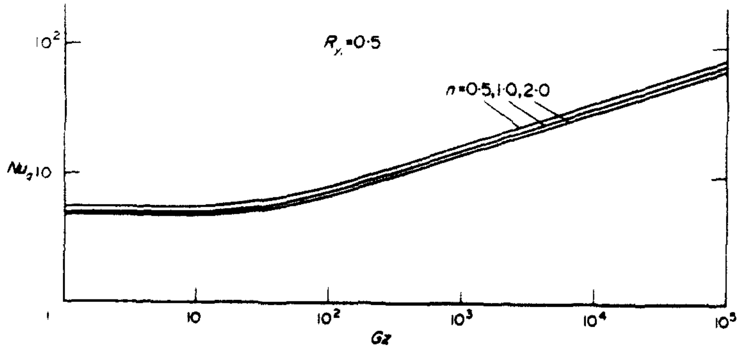

FIG. 10. Nu_q vs Gz for the heating and cooling of temperature-independent generalized Bingham plastics with constant heat flux at the tube wall.

existence of a yield stress and the change in τ_w changes the size of the flowing plug and so when $\tau_y \neq 0$ the exit velocity profile is not the same as that at the inlet to the heated section. For cooling the wall shear stress increases during heat transfer and so R_y, and hence the size of the flowing plug, decreases with decreasing Graetz number and also the velocity gradients in the wall region are decreased with decreasing Gz. These effects are shown in Fig. 6. The increase in velocity gradients in the wall region during heating enhances heat transfer while the decrease during cooling retards the heat transfer rate as shown in Figs. 7 and 8 for $R_{y_i} = 0{\cdot}5$ with $n = 0{\cdot}5$ and 2·0.

(b) *Constant wall heat flux*

For this case the heat transfer results are presented graphically as plots of Nu_q vs Gz.

Fluid consistency independent of temperature. The effect of R_{y_i} on the heat transfer results is shown in Fig. 9. As for the constant wall temperature case the increase in velocity gradients in the wall region as R_{y_i} increases, increases the heat transfer rate, and hence at a given value of Gz, the value of Nu_q is increased. Figure 10 shows the effect of n on the heat transfer results for $R_{y_i} = 0{\cdot}5$. Again, as for the constant wall temperature case, the velocity gradients in the wall region, and hence the heat transfer rate, increase as n decreases. Far downstream (Gz small) the shape of the fluid temperature profile becomes constant and the value of $(\theta_w - \theta_0)$ thus becomes constant. It can be seen from equation (42) that when this occurs, the value of Nu_q becomes a constant and some typical values are shown below.

Table 1. Asymptotic local Nusselt numbers at Gz < 1

n	R_{y_i}	Nu_q
0·5	0·5	5·604
1·0	0·5	5·136
2·0	0·5	4·811
1·0	0·7	5·863

Fluid consistency dependent on temperature. For this case the velocity profiles and the size of flowing plug change during heat transfer in a way similar to that of the constant wall temperature case. This gives an increase in heat transfer rates for heating, $\psi > 0$, and a decrease for cooling, $\psi < 0$. The development of the velocity profiles for $\psi = +0{\cdot}1$ is shown in Figs. 11 and 12 and the heat transfer results for these cases in Figs. 13 and 14. The effect of the temperature dependent fluid consistency on heat transfer is not as marked for the constant wall heat flux case as it was for the constant wall temperature

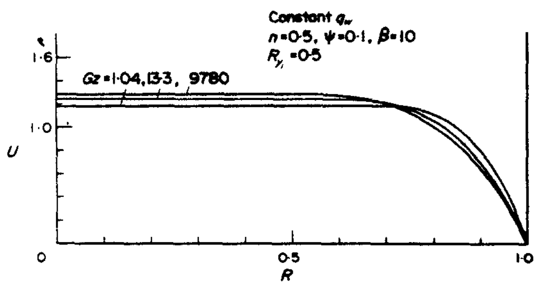

FIG. 11. Development of the velocity profile during the heating of a temperature-dependent generalized Bingham plastic (n = 0·5, τ_y/τ_w = 0·5) with constant heat flux at the tube wall.

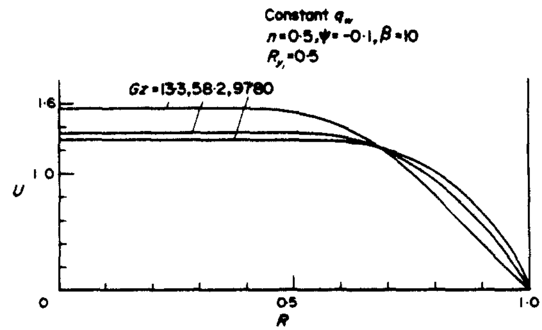

FIG. 12. Development of the velocity profile during the cooling of a temperature-dependent generalized Bingham plastic (n = 0·5, τ_y/τ_w = 0·5) with constant heat flux at the tube wall.

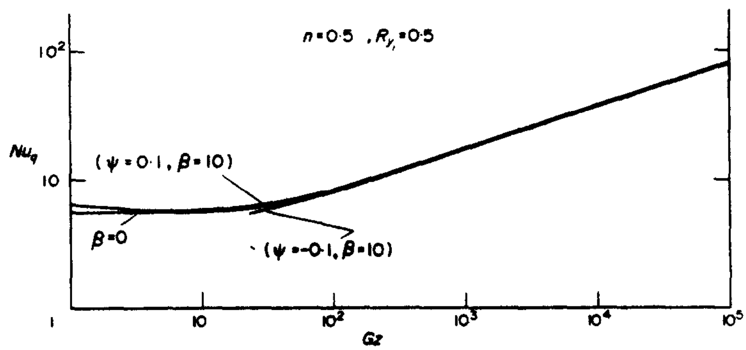

FIG. 13. Nu_q vs Gz for the heating and cooling of a temperature-dependent generalized Bingham plastic (n = 0·5, τ_y/τ_w = 0·5) with constant heat flux at the tube wall.

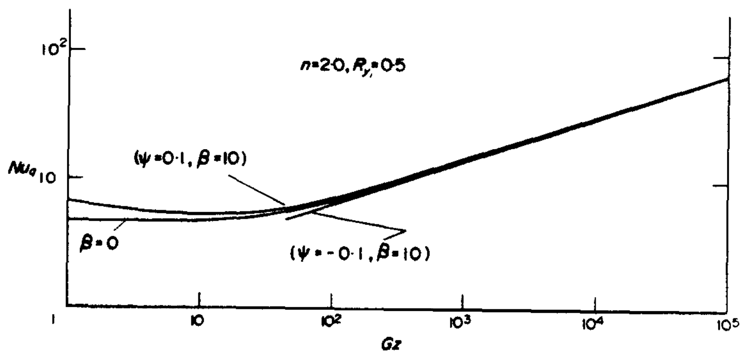

FIG. 14. Nu_q vs Gz for the heating and cooling of a temperature-dependent generalized Bingham plastic (n = 2·0, τ_y/τ_w = 0·5) with constant heat flux at the tube wall.

situation. This is because unlike the constant wall temperature case the tube wall is not subjected to a sudden large increase in temperature at the start of the heated section but instead undergoes a gradual increase under the influence of the constant wall heat flux. Thus the curves for $\beta = 0$ and $\beta = 10$ on Figs. 13 and 14 do not differ greatly.

The discontinuation of the curves for $\psi < 0$ on Figs. 13 and 14 is due to the fact that using the fluid inlet temperature as the reference temperature in equation (8) implies that the rheological model, i.e. equation (1), can not be used for situations where there is a large amount of heat removed at the wall. To cater for all conditions a new reference temperature, T_{ref}, lower than any other temperature obtained would be necessary, and hence two parameters; $q_w a/kT_{ref}$ and T_i/T_{ref}, would be required to describe the wall and inlet conditions. (N.B. for constant wall temperature these parameters would be T_w/T_{ref} and T_i/T_{ref}.)

Far downstream the value of Nu_q for $\beta \neq 0$ does not become constant since the temperature, and hence the fluid consistency and velocity profiles, are continually changing.

Viscous dissipation

When fluids which are extremely viscous are sheared, large amounts of heat can be generated due to the viscous dissipation of energy. Although this effect has not been included in this report it has been considered elsewhere [15]. For this situation the energy equation becomes

$$\rho C_p u \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \tau\left(\frac{\partial u}{\partial r}\right) \quad (43)$$

where the second term on the right represents the energy dissipation due to viscous shearing. This may be written in terms of the dimensionless variables described earlier as

$$\frac{1}{2}\frac{I_1}{I_2}\frac{\partial \theta}{\partial X} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right) + Br\left[\frac{R(R - R_y)^{1/n}(1 + \beta\theta)}{I_2^{n+1}2^{n+1}}\right] \quad (44)$$

where

$$Br = \frac{\bar{u}^{n+1}K_i}{kT_i a^{n-1}} \quad (45)$$

and represents the magnitude of the shear heating effect. Solutions to equation (44) were

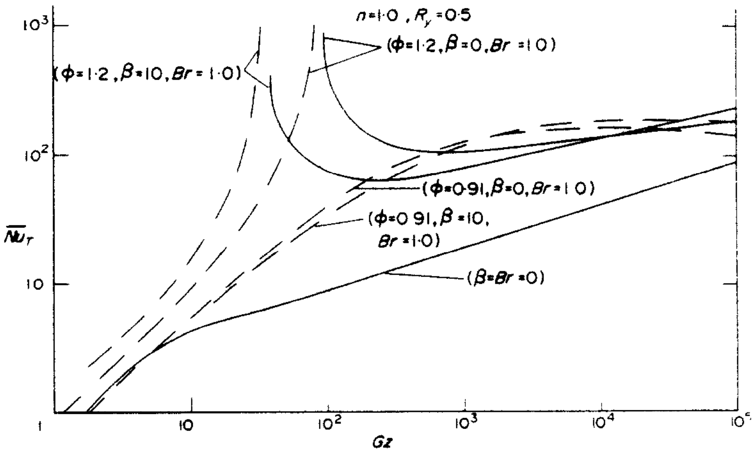

FIG. 15. $\overline{Nu}_T$ vs Gz for heat transfer to a Bingham plastic ($n = 1{\cdot}0$, $R_{y_i} = 0{\cdot}5$) when the effects of viscous dissipation are significant.

obtained for all situations where $\tau_y = 0$ and for all cases where $\tau_y > 0$ with $\beta = 0$. However, when both τ_y and β are non-zero, i.e. the fluid considered exhibited a yield stress and its consistency was dependent on temperature, solutions were not always possible at low values of the Graetz number due to instability difficulties with the numerical techniques used [15]. Some typical examples of some of the solutions that were obtained are shown in Fig. 15 in which the broken lines represent negative Nusselt numbers.

COMPARISON WITH PREVIOUS WORK

The results of the present work have been shown to be in excellent agreement with those of other workers for situations where the yield stress is zero [15]. There are little data available to check the results for situations where τ_y is not zero but some comparisons with other work have been made and are shown in Tables 2 and 3 for the case of heat transfer with a constant wall temperature. The results shown in Table 2 for situations where the fluid consistency is independent of temperature are in excellent agreement for Graetz numbers > 10. Below this value of Gz the equation proposed by Hirai [4] becomes inaccurate and thus his results no longer agree with those of Jensen [14] or of the present work.

The comparison of results shown in Table 3 is for situations where the fluid consistency is dependent on temperature. As can be seen, the results are in good agreement for $Gz < 1000$. At higher values of Gz agreement is not so good. This is probably due to the fact that the temperature dependence model used by Jensen [14] was of the Arrhenius exponential type and thus differs from the type of model used here.

Table 2. Comparison of the present work with other studies for a constant property Bingham plastic (i.e. n = 1·0) with R_{y_i} = 0·5 and constant temperature at the tube wall

Gz	$\overline{Nu}_T$ Jensen [14]	Hirai [4]	Present work
10^4	41	40·6	41
10^3	18·9	18·9	18·9
10^2	8·75	8·76	8·7
10	4·1	4·08	3·9
2	1·27	2·38	1·27

Table 3. Comparison of the present work with that of Jensen [14] for a temperature dependent Bingham plastic (i.e. n = 1·0) with R_{y_i} = 0·5 and constant temperature at the tube wall

Gz	$\overline{Nu}_T$ Jensen [14]	Present work (ϕ = 1·3 β = 63·4)
13600	94	83·3
1130	31·0	28·3
105	11·9	11·9
9·88	4·8	4·77
1·68	1·07	1·07

Gz	$\overline{Nu}_T$ Jensen [14]	Present work (ϕ = 1·2 β = 31·7)
13600	83·0	80·4
1130	30·0	29·2
105	12·1	12·1
9·88	4·80	4·80
1·68	1·07	1·07

CONCLUSIONS

The procedure which has been developed allows a detailed analysis of laminar flow heat transfer to time-independent non-Newtonian fluids in tubes to be carried out. Temperature-dependent rheological properties are included for the boundary conditions of constant tube wall temperature or constant tube wall heat flux. The solutions yield results which are functions of a number of dimensionless groups and as such should be of value in engineering design.

A selection of computed results have been presented graphically. Some interpolation is possible for engineering design purposes but more complete information is available elsewhere [15].

REFERENCES

1. N. J. BEEK and R. EGGINK, Heat transfer in laminar flow to a non-Newtonian fluid in a round tube, *De Ingeniur* **74**, 81 (1962).
2. B. C. LYCHE and R. B. BIRD, The Graetz–Nusselt problem for a power law non-Newtonian fluid, *Chem. Engng Sci.* **6**, 34 (1956).
3. I. R. WHITEMAN and W. E. DRAKE, Heat transfer to flow in a round tube with arbitrary velocity distribution, *Trans. Am. Soc. Mech. Engrs* **80**, 728 (1958).
4. E. HIRAI, Theoretical explanation of heat transfer in the laminar region of a Bingham fluid, *A.I.Ch.E. Jl.* **5** 130 (1959).
5. E. H. WISSLER and R. S. SCHECTER, The Graetz–Nusselt problem (with extension) for a Bingham plastic, *Chem. Engng Prog. Symp. Ser.* **55** (29), 203 (1959).
6. J. SCHENK and J. VAN LAAR, Heat transfer in non-Newtonian laminar flow in tubes, *J. Appl. Sci. Res.* **A7**, 449 (1958).
7. R. L. PIGFORD, Non-isothermal flow and heat transfer inside vertical tubes, *Chem. Engng Prog. Symp. Ser.* **51** (17), 79 (1955).
8. A. B. METZNER, R. D. VAUGN and G. L. HOUGHTON, Heat transfer to non-Newtonian fluids, *A.I.Ch.E. Jl.* **3**, 92 (1957).
9. A. B. METZNER and D. F. GLUCK, Heat transfer to non-Newtonian fluids under laminar flow conditions, *Chem. Engng Sci.* **12**, 185 (1960).
10. T. MIZUSHINA and Y. KURIWAKI, Heat transfer to laminar flow of pseudoplastic fluids, *Mem. Fac. Engng Kyoto Univ.* **4**, 511 (1968).
11. N. MITSUISHI and O. MIYATAKE, Heat transfer with non-Newtonian laminar flow in a tube having a constant wall heat flux, *Int. Chem. Engng* **9**, 352 (1969).
12. E. B. CHRISTIANSEN and S. E. CRAIG, Heat transfer to pseudoplastic fluids in laminar flow, *A.I.Ch.E. Jl* **8**, 154 (1962).
13. E. B. CHRISTIANSEN, G. E. JENSEN and F. S. TAO, Laminar flow heat transfer, *A.I.Ch.E. Jl* **12**, 1196 (1966).
14. G. E. JENSEN, Energy and momentum transfer to non-Newtonian fluids, Ph.D. thesis, Department of Chemical Engineering, University of Utah, U.S.A. (1963).
15. G. FORREST, Laminar heat transfer to non-Newtonian fluids, Ph.D. thesis, Department of Chemical Engineering, University of Bradford, U.K. (1972).

TRANSFERT THERMIQUE LAMINAIRE DANS LES TUBES POUR UN FLUIDE DE BINGHAM SENSIBLE A LA TEMPERATURE

Résumé—Cet article traite par voie théorique du transfert thermique dans les tubes circulaires pour un écoulement laminaire de fluide non-newtonien sensible à la température dont la relation entre la tension tangentielle τ et la vitesse de déformation $\dot{\gamma}$ est de la forme:

$$\tau = \tau_y + K(T)\,\dot{\gamma}^n$$

où τ_y est une tension de fluage. n est une constante et $K(T)$ est une fonction de la température. Ce modèle peut concerner à la fois les comportements de loi puissance et du plastique de Bingham. On considère les deux conditions aux limites de température pariétale constante et de flux thermique constant à la paroi pour les cas du chauffage et du refroidissement. Les résultats du calcul sont présentés en nombre de Nusselt fonction du nombre de Graetz à l'aide de groupes adimensionnels incluant les paramètres liés à la dépendance vis à vis de la température, aux propriétés rhéologiques et aux conditions pariétales. Cette méthode de présentation est commode pour les ingénieurs. On peut aussi déterminer les profils de température, de vitesse et les pertes de charge.

LAMINARER WÄRMEÜBERGANG AN TEMPERATURABHÄNGIGEN BINGHAM-FLUIDEN IN ROHREN

Zusammenfassung—Dieser Artikel beschreibt die theoretische Behandlung des Wärmeübergangs bei laminarer Strömung in Rohren mit kreisförmigem Querschnitt für ein nicht-Newtonisches Fluid, wofür der Zusammenhang zwischen Schubspannung τ und Änderungsrate der Schubspannung $\dot{\gamma}$ durch eine Gleichung der folgenden Form beschrieben werden kann

$$\tau = \tau_y + K(T)\dot{\gamma}^n$$

wobei τ_y die Fliessspannung, n eine Konstante und $K(T)$ eine Funktion der Temperatur ist.

Dieses Modell kann also sowohl für exponentielles als auch für Bingham-plastisches Verhalten verwendet werden. Die zwei Randbedingungen, konstante Wandtemperatur und konstanter Wärmestrom durch die Wand, sind sowohl für Heizen als auch für Kühlen behandelt. Die berechneten Ergebnisse sind dargestellt durch Auftragung der Nusselt-Zahl als Funktion der Graetz-Zahl mit dimensionslosen Gruppen, worin die temperaturabhängigen Effekte, die rheologischen Eigenschaften und die Wandbedingungen durch einen Parameter spezifiziert sind. Diese Darstellungsmethode ist für den ingenieurmässigen Bedarf geeignet. Temperaturprofile, Geschwindigkeitsprofile und Druckabfall können ebenfalls bestimmt werden.

ТЕПЛООБМЕН ПРИ ЛАМИНАРНОМ ТЕЧЕНИИ В ТРУБАХ БИНГАМОВСКИХ ЖИДКОСТЕЙ, ЗАВИСЯЩИХ ОТ ТЕМПЕРАТУРЫ

Аннотация—В статье описано теоретическое исследование теплообмена при ламинарном течении неньютоновской жидкости в круглых труьах, для которых связь между сдвиговым напряжением τ и скоростью сдвига $\dot{\gamma}$ описывается соотношением вида

$$\tau = \tau_y + K(T)\dot{\gamma}^n,$$

где τ_y—предел текучести, n—постоянная, а $K(T)$—функция температуры. Поэтому данная модель применима как для степенной, так и для пластичной бингамовской жидкости. Решаются задачи нагрева и охлаждения с граничными условиями постоянства температуры стенки и постоянства теплового потока на стенке. Результаты расчета представлены в виде графиков зависимости числа Нуссельта от числа Грэтца, куда в качестве параметров входят безразмерные комплексы, включающие температурную зависимость, реологические свойства и условия на стенке. Такой метод представления весьма удобен для решения практических инженерных задач. Он позволяет также рассчитат профили скорости, профили температуры и перепад давления.